MCWP Senior High Mathematics League

Contest 1 **October 18, 2004**

Individual Questions

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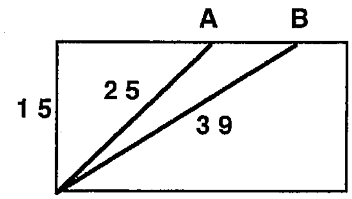
INDIVIDUAL ROUND 1 — 15 MINUTES

1. Segments drawn from a vertex of a rectangle to points A and B on the longer side of the rectangle have lengths 25 and 39, as shown. If the shorter side or the rectangle has length 15, how long is $\overline{AB}$?



2. In discussing who was the best catcher, Marina said, "Keith is best"; Ali said, "I'm not best"; Keith said, "Brian is best"; but Brian said, "Keith lied when he said I'm best". If only one of these four statements is true and the others are false, who is the best catcher ?

3. Compute the number of squares **between** 7^4 and 4^7

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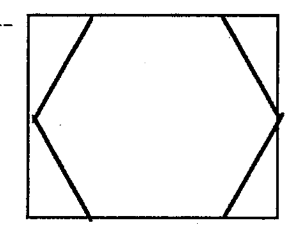
INDIVIDUAL ROUND 2 — 15 MINUTES



4. A regular hexagon whose perimeter is 12 can be inscribed in a certain rectangle, as shown. What is the area of this rectangle?

5. A line with slope -2 is 2 units from the origin. Compute the area of the triangle formed by this line and the coordinate axes.

6. (NOTE: A palindrome is a positive integer that reads backwards the same as it reads forwards. For example, 67276.)

Let S be the set of all 15-digit positive integers. An integer is chosen at random from S. The probability that the chosen integer is a palindrome is $\dfrac{1}{10^k}$. Compute the value of k.

MCWP Senior High Mathematics League

Contest #1 **October 18, 2004**

TEAM QUESTIONS

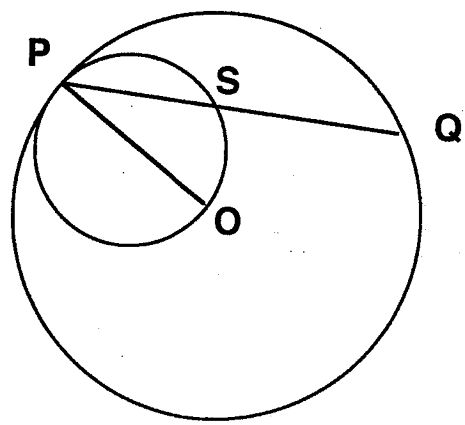
TIME LIMIT = 10 MINUTES

T-1. In a certain right triangle, if we add the lengths of the legs, and then square the sum, we'll get 1000. However, if we multiply the lengths of the legs, and then square the product, we'll get 2500. How long is the hypotenuse of this right triangle?

T-2. The solution sets of the polynomial equations, $f(x)=0$ and $g(x)=0$ are, respectively, $\{0,1,2,3\}$ and $\{2,3,4\}$. How many different numbers are in the solution set of the polynomial equation, $(f(x))(g(x))=0$?

T-3. If $A(3,4)$ and $C(7,10)$ are opposite vertices of rectangle $ABCD$, then vertices B and D must lie on the circle, $x^2+y^2-px-qy+s=0$. Compute the ordered triple of real numbers, (p,q,s).

T-4. Points P and Q are on circle O, and chord $\overline{PQ}$ is drawn. A second circle is drawn with diameter $\overline{OP}$, crossing the chord at point S. If $OP=7$ and $PQ=12$, compute PS.



T-5. There are 5 positive integers less than 6, and 3 of these are factors of 6; so 60% of the positive integers less than 6 are factors of 6. What is the smallest positive integer $n>1$ for which fewer than 1% of the positive integers less than n are factors of n?

MCWP Senior High Mathematics League

Contest #2 **INDIVIDUAL QUESTIONS** **November 15, 2004**

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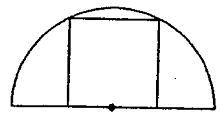
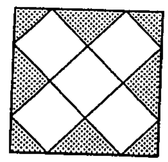
INDIVIDUAL ROUND #1 - 15 MINUTES November 15, 2004

1. The *Random House Unabridged Dictionary, 3rd edition,* correctly defines a "power" as a "product obtained by multiplying a quantity by itself one or more times. The third power of 2 is 8." How many powers of 6 lie between 100 and 2004?

2. What is the area of a square inscribed in a semircircle of radius 10?



3. The figure at the right consists of squares and isosceles right triangles. What % of the entire region is shaded?

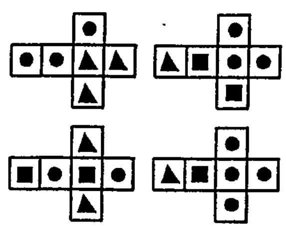


INDIVIDUAL ROUND #2 - 15 MINUTES

4. Let f be the function defined by $f(n) = n(n+1) + 41$. There are only 2 two-digit positive integers n for which ***neither*** $f(n)$ nor $f(n + 1)$ represents a prime number. What are both of these values of n?

5. What ordered pair of real numbers (x,y) satisfies $x^2 = 11 + y^2$ and $x^2 = x + y + y^2$?

6. When cutouts of the drawings at the right are properly folded and taped, four cubes are formed. On each face, a circle, a triangle, or a square is drawn, one per face, as shown. One cube is selected at random, and then one of that cube's faces is selected at random. If a circle appears on the chosen face, what is the probability that a circle also appears on the opposite face?



MCWP Senior High Mathematics League

Contest #2 **November 15, 2004**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T–1. What is the value of p for which $\sqrt{\frac{3}{8}}$ is $p\%$ of $\sqrt{6}$?

T–2. If $f(x) = |3x-1|$, compute all values of x for which $f(f(x)) = x$.

T–3. One angle of a triangle is twice another, and the sides opposite these angles have lengths 15 and 9. Find the length of the third side of the triangle.

T–4. How many digits are in the smallest positive integer whose digits have a sum of 2004?

T–5. On a 100-question test, 60 questions involve algebra and 20 questions are difficult. If 8 of the difficult questions involve algebra, how many of the questions that are ***not*** difficult ***do not*** involve algebra?

MCWP Senior High Mathematics League

Contest #3 **INDIVIDUAL QUESTIONS** **December 13, 2004**

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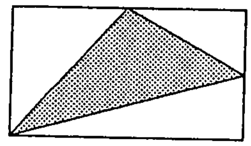
INDIVIDUAL ROUND #1 - 15 MINUTES December 13, 2004

1. The odd number k satisfies $2k-2 = 4m$, where $k-m = 502$. What is the value of $4m$?

2. The vertices of a triangle, shown shaded, are one vertex of a rectangle and midpoints of the two sides furthest from this vertex. What fractional part of the area of the rectangle is the area of the shaded triangle?



3. Regular polygon $ABCDE\ldots$ has x sides. What is the degree-measure of the acute angle at which the extensions of sides $\overline{AB}$ and $\overline{CD}$ would meet? (Write your answer in terms of x.)

INDIVIDUAL ROUND #2 - 15 MINUTES

4. What are all ordered pairs of real numbers (x,y) for which $x^2 = 7x+3y$ and $y^2 = 3x+7y$?

5. If $a \neq b$ and $ax + by = bx + ay$, what is the value of $x - y$?

6. Into at most how many distinct, disjoint regions can three triangles partition a plane?

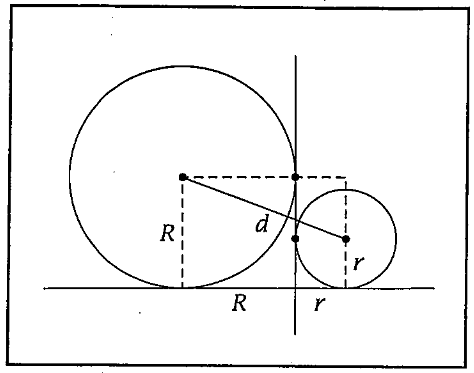
MCWP Senior High Mathematics League

Contest #3 **December 13, 2004**

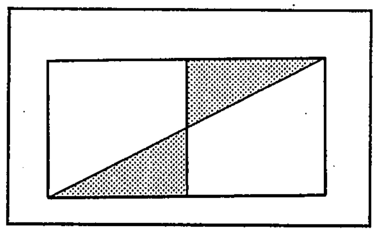
TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T–1. Two circles have radii 1 and 7, and one of their common internal tangents is perpendicular to a common external tangent. Compute the distance between their centers.



T–2. A line segment is drawn connecting the midpoints of two opposite sides of a rectangle, as shown in the diagram. A diagonal of the rectangle is also drawn. If the area of each shaded triangle is 20, what is the area of the original rectangle?



T–3. The median of a set of numbers is the arithmetic mean of the two middle numbers. Given the set of numbers, $\left\{2^k, 2^{k+1}, 2^{k+2}, 2^{k+3}\right\}$, if k is a positive integer, what is the result, in simplest form, of dividing the median of this set by the smallest number in the set?

T–4. By finding a gold nugget, a prospector is said to "strike it rich". If, on a given day, a prospector has probability $\frac{1}{8}$ of "striking it rich", what is the probability that at least one of the prospectors, Bill and Barb, "strike it rich" on a day on which they both mine for gold?

T–5. Find the value of the integer x that satisfies the equation,

$$\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{8^2}\right)\cdots\left(1-\frac{1}{2004^2}\right)=\frac{x}{2004}.$$

MCWP Senior High Mathematics League

Contest #4 **INDIVIDUAL QUESTIONS** **January 10, 2005**

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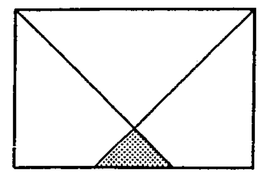
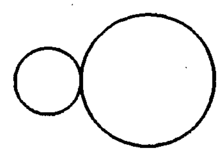
INDIVIDUAL ROUND #1 - 15 MINUTES January 10, 2005

1. One stamp is randomly selected from a 10×10 sheet of 100 stamps. What is the probability that the stamp was *not* one of the sheet's border stamps.

2. Each trisection point of one side of a rectangle is connected, by a line segment, to the vertex of the rectangle furthest from it, as shown at the right. What fractional part of the rectangle is occupied by the shaded triangle?



3. One circle has half the circumference of another. How many rotations about its own center will the smaller circle make in rolling exactly once around the larger one, in the manner shown?



INDIVIDUAL ROUND #2 - 15 MINUTES

4. What value of x satisfies $4^x + 4^{1-x} = 4$?

5. A deck with 26 red and 26 black cards is separated into two unequal nonempty piles. The first pile contains 7 times as many black cards as red. The number of red cards in the second pile is a multiple of the number of black cards in that pile. How many red cards are in the first pile?

6. What is the radian measure of the least positive angle x for which $\tan(5x+x) = \dfrac{\cos x + \sin x}{\cos x - \sin x}$?

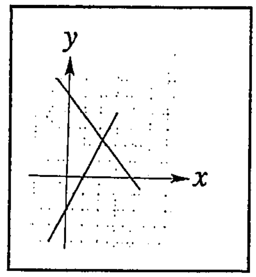
MCWP Senior High Mathematics League

Contest #4 **January 10, 2005**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T–1. In the graph shown, one line segment passes through $(0,5)$ and $(4,0)$, and the other passes through $(0,-2)$ and $(1,0)$. What are the coordinates of the point of intersection of these two segments?



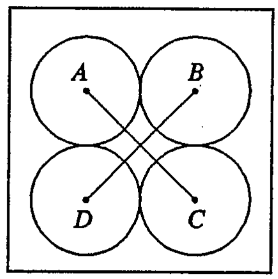
T–2. (NOTE: A palindrome is a positive integer that reads backwards the same as it reads forwards. For example, 67276.)

John thought that he had added together all the 2-digit positive integers, and the sum he got was a palindrome. Unfortunately, he had left one number out. What number had been omitted?

T–3. If n is a 3-digit positive integer, and $n = ABC = AB + BA + AC + CA + BC + CB$, compute the largest possible value for n.

T–4. A group of students stood evenly spaced in a circular formation. They counted off, starting at 1 and continuing by consecutive integers around the circle, clockwise. After all students had counted off, the 19th student noticed that the student farthest from him was student 99. How many students were in the entire group?

T–5. A, B, C, and D are the centers of four congruent circles which are tangent to each other as shown. If $AC = BD = 12$, what is the area of one of the four circles?



MCWP Senior High Mathematics League

Contest #5 INDIVIDUAL QUESTIONS February 7, 2005

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INDIVIDUAL ROUND #1 - 15 MINUTES February 7, 2005

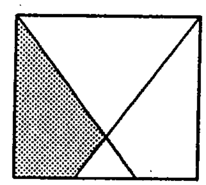
1. A sealed box with dimensions 2, 3, and 4 units is filled with 20 cubic units of water. Which dimension should be chosen as the height of the box so that the actual height of the water inside the closed box is minimized?

2. What are all real numbers x which satisfy $\frac{1}{x^2(x^2+1)+1} + \frac{1}{x^2(x^2+2)+1} = 3$?

3. If $a > 0$ and $b > 0$, what is the least possible value of $(a+b)(\frac{1}{a}+\frac{1}{b})$?

INDIVIDUAL ROUND #2 - 15 MINUTES

4. Each trisection point of one side of a rectangle is connected, by a line segment, to the vertex of the rectangle furthest from it, as shown at the right. What fractional part of the area of the rectangle is the area of the shaded quadrilateral?



5. Reduce $\frac{1\times 4\times 9\times 16\times ...\times n^2\times ...\times 10000}{2\times 6\times 12\times 20\times ...\times (n)(n+1)\times ...\times 10100}$ to lowest terms.

6. What is the numerical value of b for which the length of the path from $A(0,2)$ to $B(b,0)$ to $C(c,10)$ to $D(5,9)$ will be a minimum?

MCWP Senior High Mathematics League

Contest #5 **February 7, 2005**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T–1. In a convex polygon of n sides, one interior angle contains $x°$, while each of the remaining $n-1$ interior angles contains $133°$. Compute all possible values of $\boldsymbol{x}$.

T–2. Find the real value x which satisfies the equation, $\left(x^2+x+1\right)\left(x^6+x^3+1\right)=\dfrac{10}{x-1}$.

T–3. What is the least integer n greater than 1 for which the square root of n, the cube root of n, and the fourth root of n are all integers?

T–4. Three solid gold spherical balls have diameters of 3, 4, and 5 cm. If the three balls of gold are melted down and recast into a single solid gold spherical ball of diameter $\boldsymbol{d}$, what is the value of $\boldsymbol{d}$?

T–5. If x is randomly chosen from the positive real numbers less than 10, what is the probability that $x+\dfrac{3}{x}\leq 4$?

MCWP Senior High Mathematics League

Contest #6 **INDIVIDUAL QUESTIONS** **March 7, 2005**

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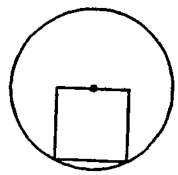
INDIVIDUAL ROUND #1 - 15 MINUTES March 7, 2005

1. One side of a square is a chord of a circle. The opposite side of the square passes through the center of the circle, as shown. If the area of the circle is 100π, what is the area of the square?

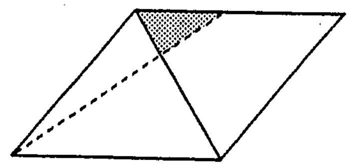


2. What is the only complex number $u+vi$ for which there does **NOT** exist a complex number $x+yi$ such that $ux-vy = 1$ and $uy+vx = 0$? [In this problem, u, v, x, and y are real numbers.]

3. What is the least possible degree of a polynomial equation with integer coefficients two of whose roots are $1+\sqrt{3}$ and $2-\sqrt{5}$?

INDIVIDUAL ROUND #2 - 15 MINUTES

4. The dotted segment divides a diagonal of the parallelogram in the ratio 3 to 7, as shown. What fractional part of the area of the parallelogram is the area of the shaded triangle?



5. What are all values of x for which $\frac{x-1}{x+2} \div \frac{x-3}{x+4}$ becomes meaningless?

6. In a single-knockout, elimination-type tournament, each match pits 2 contestants against each other. The winner of a match continues to play further matches, while the loser is eliminated from further competition. The matches continue until an overall winner is determined. If 100 players originally enter the tournament, and if every match has one winner, how many matches must be played in order for an overall winner to be determined? [Note that a bye round, in which a player does not actually compete, but is permitted to continue to play in further matches, does not count as a match.]

MCWP Senior High Mathematics League

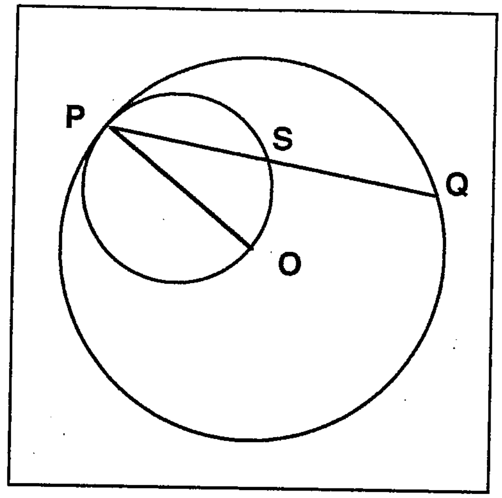
Contest #6 **March 7, 2005**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T–1. Let $P(x) = x^3 + 6x^2 + 7x + 2$. What is the real number c for which the polynomial $P(x+c)$ has no x^2 term?

T–2. Points P and Q are on circle O, and chord $\overline{PQ}$ is drawn. A second circle is drawn with diameter $\overline{OP}$, crossing the chord at point S. If $OP = 7$ and $PQ = 12$, compute PS.

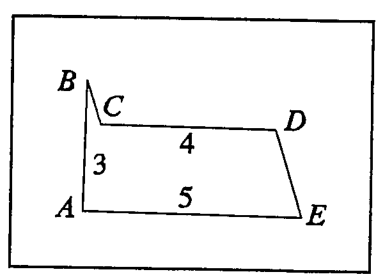


T–3. If a is an integer, and if both roots of $x^2 + ax + 17 = 0$ are positive integers, what is the value of a?

T–4. **NOTE:** $\binom{a}{b}$ represents the number of combinations of a things taken b at a time.

Find the smallest positive integer $n > 100$ so that $\binom{n}{101}$ is divisible by $\binom{n}{100}$, but is not equal to it.

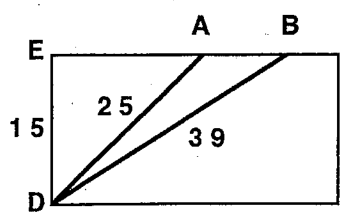
T–5. In the diagram, $\overline{AE} \perp \overline{AB}$, $\overline{CD} \parallel \overline{AE}$, and $\overline{BC} \parallel \overline{DE}$. If $CD = 4$, $AB = 3$, and $AE = 5$, and the distance from $\overline{AE}$ to $\overline{CD}$ is 2, what is the area of pentagon $ABCDE$?



SOLUTIONS TO INDIVIDUAL QUESTIONS

Contest #1, 2004-2005 **October 18, 2004**

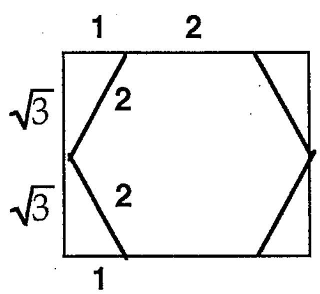
I-1. The width of the rectangle is 15, so right triangle ΔDEA and right triangle ΔDEB each have a leg of length 15. By the Pythagorean Theorem, $EA = 20$ and $EB = 36$. Therefore, $AB = 36 - 20 = 16$.



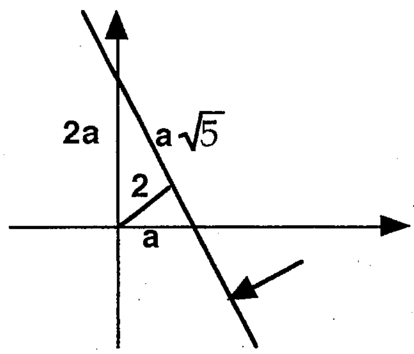
I-2. If Brian were best, then Ali and Keith both told the truth. If Marina were best, then Ali and Brian both told the truth. If Keith were best, then everyone except Keith told the truth. Finally, if Ali were best, then only Brian told the truth. So, Ali is best.

I-3. Write 7^4 as 49^2, and write 4^7 as $\left(2^7\right)^2 = 128^2$. Then, we are looking for the number of squares between 49^2 and 128^2. The number of such squares is $128 - 49 - 1 = 78$.

I-4. When the hexagon is inscribed in the rectangle as shown, four 30°–60°–90° triangles are formed in the corners of the rectangle. These triangles have dimensions $1, 2, \sqrt{3}$ as shown. Since the rectangle has a width of 4, and a height of $2\sqrt{3}$, its area is $8\sqrt{3}$.



I-5. Since the slope of the line is -2, we can let the legs of the triangle be a and $2a$. Then the area is of the triangle is given by: $\frac{1}{2}(2a)(a) = a^2$. Since the hypotenuse has length $a\sqrt{5}$, and since the altitude to the hypotenuse is given as 2, the area is also given by: $\frac{1}{2}\left(a\sqrt{5}\right)(2) = a\sqrt{5}$. Equating these two expressions, we have $a = \sqrt{5}$, so the area of the triangle is 5.



I-6. To form a 15-digit palindrome, there are 9 choices for the first digit, 10 choices for each of the digits in the next 7 places, and the last 7 digits must match the first 7 digits. Therefore, the number of 15-digit palindromes is $(9)\left(10^7\right)$. Set S contains $(9)\left(10^{14}\right)$ 15-digit integers. So, the probability of choosing a palindrome, is $\dfrac{(9)\left(10^7\right)}{(9)\left(10^{14}\right)} = \dfrac{1}{10^7}$, and $k = 7$.

SOLUTIONS TO TEAM QUESTIONS

Contest #1, 2004 - 2005 **October 18, 2004**

T-1. Let the lengths of the legs be a and b, with $a > b$. We are given that $(a+b)^2 = 1000$, and that $(ab)^2 = 2500$. Squaring in the first equation, we have $a^2 + 2ab + b^2 = 1000$. From the second equation, we have $ab = 50$. Substituting into the first equation, we have $a^2 + 2(50) + b^2 = 1000$, from which we can see that $a^2 + b^2 = 900$. Let c be the length of the hypotenuse. Then, $c^2 = a^2 + b^2 = 900$, we have $c = 30$.

T-2. When polynomial equations are multiplied together, all factors of either original equation are factors of the product, and vice versa. Therefore, in this problem, the solution set of the new equation is $\{0,1,2,3,4\}$, the union of the original two solution sets. Therefore, the number of solutions is 5.

T-3. The center of the circle is the midpoint of the diagonal of the rectangle. Therefore the circle has center $C(5,7)$, and its radius is $\sqrt{13}$. Then the equation of the circle is:

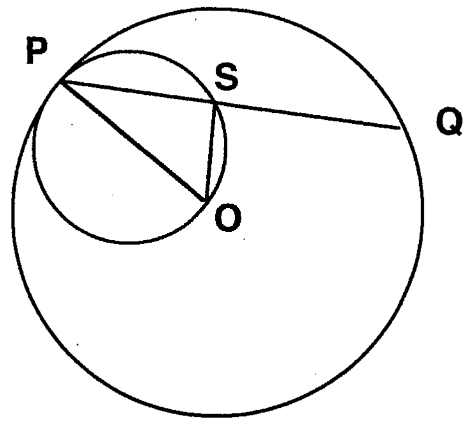
$(x-5)^2 + (y-7)^2 = \left(\sqrt{13}\right)^2$ or

$x^2 + y^2 - 10x - 14y + 61 = 0$

So, the required ordered triple is:
(10,14,61)

T-4. Since $\angle OSP$ is a right angle, $\overline{OS}$ is perpendicular to $\overline{PQ}$. Therefore, it bisects $\overline{PQ}$, so $PS = 6$.

(NOTE: The length of $\overline{OP}$ does not matter!)



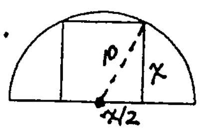
T-5. Since fewer than 1% of the positive integers less than n are factors of n, there must be more than 100 integers less than n that are not factors of n. Therefore, $n > 101$. We can quickly see that the required answer is 103, since this is the least prime number greater than 101.

Math Council of Western PA Sr High Math Leauge 2004-2005
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #2 — November 15, 2004

Answers

1. 2
2. 80
3. 37.5 or 37.5%
4. 40, 81
5. (6,5)
6. 1/3

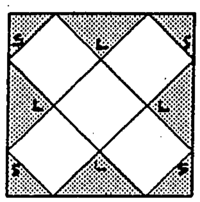
1. $6^0=1;\ 6^1=6;\ 6^2=36;\ \underbrace{6^3=216;\ 6^4=1296}_{\text{these are the only } \boxed{2}};\ 6^5>2004$

2. 

$x^2+\left(\frac{x}{2}\right)^2=100$

$5x^2=400$

$x^2=\boxed{80}$.

3. 

Each small isos. rt. $\Delta = 1/4$ sq. unit in area. (S)
Each large isos. rt. $\Delta = 1/2$ sq. unit in area (L)
4 small + 4 large $= 4(\frac{1}{4})+4(\frac{1}{2}) = 3$ sq. units of area
unshaded region $= 5$ sq. units of area.
Total area $= 8$, 3 of which are shaded. Finally, $\frac{3}{8} = \underline{37.5\%}$

4. When

n =	40	41	81	82
n+1 =	41	42	82	83

41 is a factor of $n(n+1)+41$.
Thus, $n = \boxed{40, 81}$.

5. $x^2 = 11 + y^2 \rightarrow x^2 - y^2 = (x+y)(x-y) = 11$

$x^2 = x+y+y^2$

$0 = 11-x-y$

$\underline{x+y=11}$

$11(x-y)=11$

$\underline{x-y=1}$

$x+y=11$
$x-y=1$
$2x=12$
$x=6,\ y=5$
$(x,y)=\boxed{(6,5)}$.

6. Since we are told that the face that was chosen contains a circle, it's twice as likely that the face came from the lower right cube, which contains 4 circles, than from the lower left cube, which contains 2 circles. The probability $= \dfrac{3\times\frac{0}{3}+3\times\frac{0}{3}+2\times\frac{2}{2}+4\times\frac{2}{4}}{12} = \boxed{\frac{1}{3}}$.

Method II: There are 12 circles. Of these 4 have circles across from them, so $P = 4/12 = 1/3$.

Math Council of Western PA Sr. High Math League 2004 - 2005

Contest # 2 — SOLUTIONS TO TEAM ROUND **November 15, 2004**

T–1. We have, $\frac{p}{100}\left(\sqrt{6}\right)=\sqrt{\frac{3}{8}}$, so $p=\frac{100\sqrt{3}}{\sqrt{6}\sqrt{8}}=\frac{100}{\sqrt{16}}=25$

NOTE: The value of p is 25, not 25%. (Read the problem statement carefully!)

==========

T–2. $f(f(x))=\left|\,3\,|3x-1|\,-1\,\right|=x$ implies $3\,|3x-1|-1=\pm x$, so $3\,|3x-1|=1\pm x$.

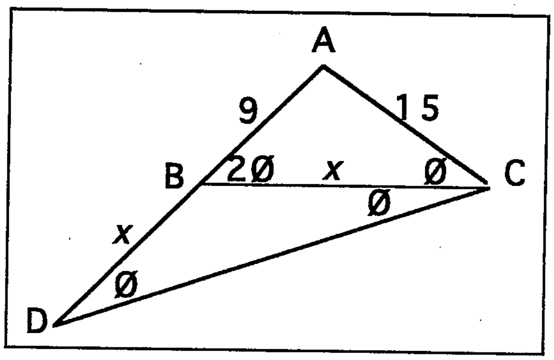
If $x\geq\frac{1}{3}$, we have $9x-3=1\pm x$, leading to $x=\frac{1}{2}$ *or* $\frac{2}{5}$, both of which check.

If $x<\frac{1}{3}$, we have $3-9x=1\pm x$, leading to $x=\frac{1}{5}$ *or* $\frac{1}{4}$, both of which check.

So, the answers are: $\frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{1}{2}$

==========

T–3. Let $m\angle ACB=\emptyset$. Then $m\angle ABC=2\emptyset$. Extend side AB through B to D so that $BD=BC=x$. Then ΔCBD is isosceles and $m\angle BCD=m\angle BDC$. We can also see by the exterior angle theorem, that $m\angle BCD+m\angle BDC=m\angle ABC=2\emptyset$. So, $m\angle BCD=m\angle BDC=\emptyset$. Therefore, ΔADC is similar to ΔACB and we have the proportion, $\frac{9+x}{15}=\frac{15}{9}$, from which we find that $x=16$.



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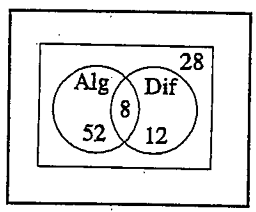
T–4. The smallest such positive integer will have as few digits as possible, so all but the first digit must be 9's. Since $9\times 222=1998$, the integer we are seeking is $6999\ldots99$ — in other words, the digit 6 followed by 222 nines. This number has 223 digits.

==========

T–5. There are 8 difficult algebra questions, so there are $60-8=52$ non-difficult algebra questions. Since there are 80 non-difficult questions, and 52 of these involve algebra, the number of questions that are neither difficult nor involve algebra is $80-52=28$.



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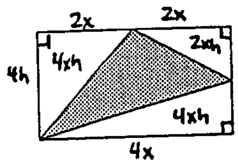
Math Council of Western PA Sr High Math Leauge 2004 - 2005
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #3 — December 13, 2004

Answers

1. 2004
2. 3/8 or 0.375
3. $180 - (720/x)$ or $180(x-4)/x$ or exact equivalent
4. (6,−2), (−2,6), (10,10), (0,0)
5. 0
6. 20

1. $2k-2=4m \Leftrightarrow 2k-2m=2m+2$
$$2(k-m)=2m+2$$
$$2(502)=2m+2=1004$$
$$2m=1002$$
$$4m=\boxed{2004}.$$

2.

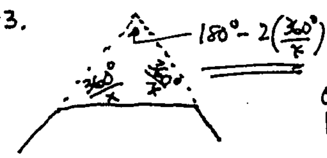


Area of rectangle $=16xh$

Subtract unshaded triangle's areas.

$$16xh-4xh-2xh-4xh=6xh$$
$$\frac{6xh}{16xh}=\boxed{\frac{3}{8}}.$$

3.



each exterior ∡ of a regular polygon has a measure of $360/x$.

4. Subtracting $x^2-y^2=4(x-y)$, or $(x+y)(x-y)=4(x-y)$. Thus, either $x=y$ or $x+y=4$. If $x=y$, then $x^2=7x+3x=10x$, so $x=0$ or 10. If

If $x=4-y$, then $x^2=7x+3(4-x)=4x+12$, so $x^2-4x-12=(x-6)(x+2)=0$, $x=6$ or $x=-2$

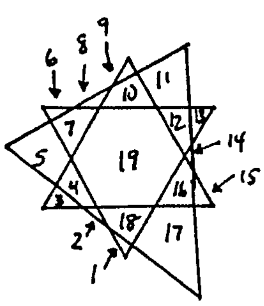
There are 4 solutions:

$\boxed{(6,-2),(-2,6),(10,10),(0,0)}$

5. $ax-ay=bx-by \Leftrightarrow a(x-y)=b(x-y) \Leftrightarrow (a-b)(x-y)=0$

Since $(a-b)\neq 0$, $x-y=\boxed{0}$.

6.



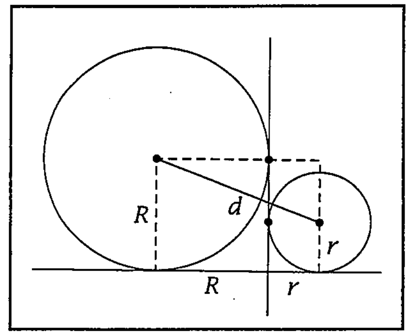
The exterior is also a region, so the total # is $\boxed{20}$.

Every side of each triangle crosses two sides of every other triangle, for max. # of regions

Math Council of Western PA Sr. High Math League 2004 - 2005

Contest # 3 — SOLUTIONS TO TEAM ROUND **December 13, 2004**

T–1. We have $d^2 = (R+r)^2 + (R-r)^2 = 2(R^2 + r^2) = 2(49+1) = 100$, so $d = 10$.



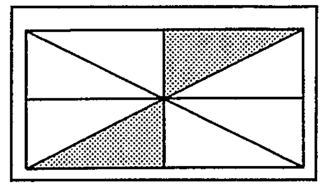
T–2.

METHOD 1:

The rectangle consists of 8 congruent triangles, so the area of the rectangle is $8 \times 20 = 160$

METHOD 2:

Since the length of each triangle is half the length L of the rectangle, and the width of each triangle is half the width W of the rectangle, the area of each triangle is $\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{W}{2}\right) = \frac{LW}{8}$. Therefore, $\frac{LW}{8} = 20$ and $LW = 160$



T–3. The median is $\frac{\left(2^{k+1} + 2^{k+2}\right)}{2} = 2^k + 2^{k+1}$.

Dividing this by the smallest number, 2^k, we get $\frac{2^k + 2^{k+1}}{2^k} = 1 + 2 = 3$

T–4. The probability that either strikes it rich (and therefore, wins) is $\frac{1}{8}$, so the probability that either one does not strike it rich (loses) is $\frac{7}{8}$.

Since $P(Bill\ wins, Barb\ loses) = P(Barb\ wins, Bill\ loses) = \frac{1}{8} \times \frac{7}{8} = \frac{7}{64}$, and since $P(both\ win) = \frac{1}{8} \times \frac{1}{8} = \frac{1}{64}$, the probability that at least one of them wins is $\frac{7}{64} + \frac{7}{64} + \frac{1}{64} = \frac{15}{64}$

T–5. The left side can be written, $\frac{5^2-1}{5^2} \bullet \frac{6^2-1}{6^2} \bullet \frac{7^2-1}{7^2} \bullet \frac{8^2-1}{8^2} \bullet\bullet\bullet \frac{2003^2-1}{2003^2} \bullet \frac{2004^2-1}{2004^2}$

$$= \frac{(5-1)(5+1)}{(5)(5)} \bullet \frac{(6-1)(6+1)}{(6)(6)} \bullet \frac{(7-1)(7+1)}{(7)(7)} \bullet\bullet\bullet \frac{(2003-1)(2003+1)}{(2003)(2003)} \bullet \frac{(2004-1)(2004+1)}{(2004)(2004)}$$

$$= \frac{(4)\cancel{(6)}}{(5)\cancel{(5)}} \bullet \frac{\cancel{(5)}\cancel{(7)}}{\cancel{(6)}\cancel{(6)}} \bullet \frac{\cancel{(6)}\cancel{(8)}}{\cancel{(7)}\cancel{(7)}} \bullet \frac{\cancel{(7)}\cancel{(9)}}{\cancel{(8)}\cancel{(8)}} \bullet \frac{\cancel{(8)}\cancel{(10)}}{\cancel{(9)}\cancel{(9)}} \bullet\bullet\bullet \frac{\cancel{(2002)}\cancel{(2004)}}{\cancel{(2003)}\cancel{(2003)}} \bullet \frac{\cancel{(2003)}(2005)}{\cancel{(2004)}(2004)}$$

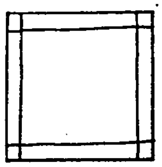
$$= \frac{4}{5} \bullet \frac{2005}{2004} = \frac{4(401)}{2004}.$$ Therefore, $x = 1604$.

Math Council of Western PA Sr High Math Leauge 2004 - 2005
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #4 — January 10, 2005

Answers

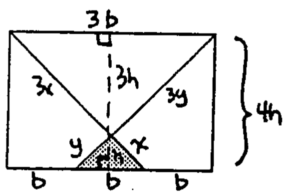
1. 64% or 0.64 or 16/25
2. 1/24
3. 3
4. 1/2 or 0.5
5. 2
6. $\pi/20$

1.



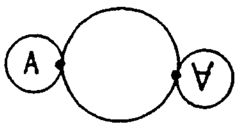
the borders overlap in 4 places
The total # of border stamps is $4(10) - 4 = 36$
$\text{Prob(not a border stamp)} = 1 - P(\text{border stamp})$
$= 1 - \frac{36}{100} = \boxed{0.64}$.

2.



The area of the rectangle is $(3b)(4h) = 12bh$
Area of shaded $\Delta = \frac{bh}{2}$.
Fractional part $= \frac{bh/2}{12bh} = \boxed{\frac{1}{24}}$.

3.



When the small circle rolls one of its circumferences, it will be on the right, upside down, as seen. It has gone half-way around, yet made 1 1/2 rotations

4. Let $y = 4^x$. Then $\frac{1}{y} = 4^{-x}$ and $\frac{4}{y} = 4^{1-x}$. Thus, $y + \frac{4}{y} = 4$, so
$y^2 - 4y + 4 = 0$. Thus,
$y = 2$ and $x = \boxed{1/2}$.

5.

1st pile Red	1st pile Black	2nd pile Red	2nd pile Black	
1	7	25	19	NO
$\boxed{2}$	14	24	12 ← Aha!	
3	21	23	5	NO

(no other possibilities)

6. $\tan(5x + x) = \frac{\cos x + \sin x}{\cos x - \sin x} = \frac{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}} = \frac{1 + \tan x}{1 - \tan x} = \frac{\tan 5x + \tan x}{1 - \tan 5x \tan x}$

Thus, $\tan 5x = 1$, $5x = \frac{\pi}{4}$,
and $x = \boxed{\frac{\pi}{20}}$, at least.

Math Council of Western PA Sr. High Math League 2004 - 2005

Contest # 4 — SOLUTIONS TO TEAM ROUND January 10, 2005

T–1. The lines have equations $y=2x-2$ and $y=-\frac{5}{4}x+5$, so the lines intersect where $2x-2=-\frac{5}{4}x+5$. The point of intersection is $\left(\dfrac{28}{13},\dfrac{30}{13}\right)$

==

T–2. The correct sum is $10+11+\cdots+99=\dfrac{99(100)}{2}-\dfrac{9(10)}{2}=4905$. John's palindrome must have been 4,884, so the omitted number was 21.

==

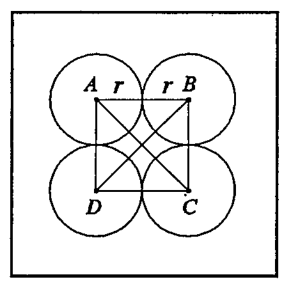
T–3. We have $100A+10B+C$
$=(10A+B)+(10B+A)+(10A+C)+(10C+A)+(10B+C)+(10C+B)$
$=22A+22B+22C$. So, $100A+10B+C=22(A+B+C)$. This leads to $26A=4B+7C$. Since $4B+7C$ cannot exceed $4\times9+7\times9=99$, A cannot exceed 3. Trying A = 3, we quickly see that B = 9 and C = 6 lead to the greatest possible value, n = 396.

==

T–4. Students numbered 20 through 98 (79 in all) are between student 19 and student 99. Similarly, counting in the other direction, there are 79 students between student 99 and student 19. If we then also include student 19 and student 99, we find the total number of students to be: $79+79+1+1=160$.

==

T–5. Since $ABCD$ is a square whose diagonals are each 12, $AB=6\sqrt{2}$. This distance represents $2r$, so $r=3\sqrt{2}$ and the area of one circle is $\pi\left(3\sqrt{2}\right)^2=18\pi$



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Math Council of Western PA Sr High Math Leauge 2004 - 2005
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #5 — February 7, 2005

Answers

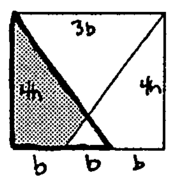
1. 2
2. *none* or { } or ∅ or *empty set* Do *not* accept {∅}.
3. 4
4. 7/24
5. 1/101
6. 10/13

1. Since $4 - \frac{20}{2\cdot3} > 3 - \frac{20}{2\cdot4} > \boxed{2} - \frac{20}{3\cdot4}$, so make the shortest dimension become the "height" of the box.

2. If x is real, then $x^2 \ge 0$, so $x^2+1 > 0$ and $x^2+2>0$. Hence, $x^2(x+2)+1 \ge x^2(x+1)+1 \ge 1$. Thus, their reciprocals are ≤ 1, so their sum is a positive number ≤ 2. $\boxed{\text{No solutions}}$ are possible in the reals.
Alternatively, a graphical representation establishes this result too.

3. $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = 2 + \frac{a}{b} + \frac{b}{a}$. Let $r = \frac{a}{b}$. Then $\frac{1}{r} = \frac{b}{a}$, so we get $2 + r + \frac{1}{r}$. Since, for real numbers, if $r>0$, $2+r+\frac{1}{r} \ge 2+2 \ge 4$. When $a=b=1$, the minimum value of $\boxed{4}$ is achieved.

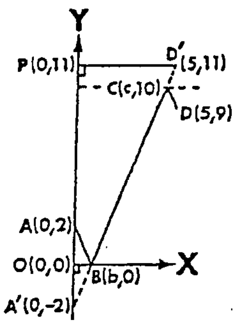
4.



Area of rectangle $= 12bh$
Area of triangle outlined in thick ink $= 4bh = \frac{1}{3}$ rectangle
On previous contest, little unshaded $\Delta = \frac{1}{24}$ rectangle.
Thus, shaded region $= \frac{8}{24} - \frac{1}{24} = \boxed{\frac{7}{24}}$ of area of rectangle.

5. $$\frac{1\times4\times9\times16\times\ldots\times n^2\times\ldots\times10000}{2\times6\times12\times20\times\ldots\times n(n+1)\times\ldots\times10100} = \frac{1\times\cancel{2}\times\cancel{2}\times\cancel{3}\times\cancel{3}\times\cancel{4}\times\cancel{4}\times\ldots\times\cancel{n}\times\cancel{n}\times\ldots\times\cancel{100}\times\cancel{100}}{\cancel{2}\times\cancel{2}\times\cancel{3}\times\cancel{3}\times\cancel{4}\times\cancel{4}\times\cancel{5}\times\ldots\times\cancel{n}(\cancel{n+1})\times\ldots\times\cancel{100}\times101} = \boxed{\frac{1}{101}}.$$

6.



$\overline{A'D'}$ is the path.
$\Delta A'OB \sim \Delta A'PD'$, so $\frac{2}{b} = \frac{13}{5}$, and $b = \boxed{\frac{10}{13}}$.

Math Council of Western PA Sr. High Math League 2004 - 2005

Contest # 5 — SOLUTIONS TO TEAM ROUND February 7, 2005

T–1. The equation $133(n-1)+x=180(n-2)$ implies $x=47n-227$. Since $0° < x < 180°$, we can only have $n=5,\ 6,\ 7,\ 8,$ producing $x=8,\ 55,\ 102,\ 149$.

T–2. We have $(x-1)(x^2+x+1)(x^6+x^3+1)=(x^3-1)(x^6+x^3+1)=x^9-1=10$. This implies that $x^9=11$, so $x=\sqrt[9]{11}$

T–3. Since n is a perfect square, cube, and fourth power, and since $n>1$, we conclude that the least such integral $n=2^x$, where x is the least common multiple of 2, 3, and 4. Therefore, $x=12$ and $2^{12}=4096$.

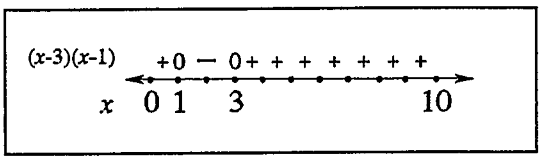
T–4.

METHOD 1: The volume of a sphere is proportional to the cube of its diameter. If we divide through by the constant of proportionality, we get $d^3=3^3+4^3+5^3=216$ so $d=6$.

METHOD 2: Let the diameters of the small gold balls be $2a, 2b,$ and $2c$. Then the total volume of the 3 small gold balls is $\frac{4}{3}\pi\left(a^3+b^3+c^3\right)=\frac{4}{3}\pi\left(\left(\frac{3}{2}\right)^3+2^3+\left(\frac{5}{2}\right)^3\right)=\frac{4}{3}\pi(27)$.

If R is the radius of the final gold ball, its volume is $\frac{4}{3}\pi R^3$, so $R=3$ and the diameter is 6.

T–5. Since $x>0$, we can multiply through by x without changing the direction of the inequality. When we do this and combine terms, we have $x^2-4x+3=(x-3)(x-1)\le 0$. Using the number line, we find that this inequality is true when $1\le x\le 3$, an interval of length 2. Therefore, the probability is $\frac{2}{10}$ that a randomly chosen positive real number less than 10 will satisfy the given condition.

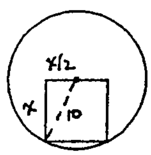


Math Council of Western PA Sr High Math Leauge 2004 - 2005
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #6 — March 7, 2005

Answers

1. 80
2. 0 or $0+0i$
3. 4
4. 9/140
5. −4, −2, 3
6. 99

1.



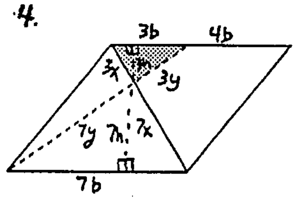
$x^2+\left(\frac{x}{2}\right)^2=100$
$5x^2=400$
$x^2=\boxed{80}$

This is IDENTICAL to problem 1-2.

2. The given conditions are equivalent to $(u+vi)(x+yi)=1$. For any $u+vi\neq 0$, there is such an $x+yi$. For $u+vi=0$, there is no such $x+yi$. So, there is one complex number $u+vi$ of the required form, namely $u+vi=0+0i=\boxed{0}$.

3. If $1+\sqrt{3}$ is a root of a polynomial equation with integer coefficients, then $1-\sqrt{3}$ is also root. Similarly, $2-\sqrt{5}$ and $2+\sqrt{5}$ are both roots. Since there are 4 roots, the least possible degree is 4. In fact, one such equation is $(x-(1-\sqrt{5}))(x-(1+\sqrt{3}))(x-(2-\sqrt{5}))(x-(2+\sqrt{5}))=0$ reduces to
$$x^4-6x^3+5x^2+10x+2=0$$

4.



Shaded region's area $=\frac{(3b)(3h)}{2}=\frac{9bh}{2}$
Area of ▱ $=(7b)(10h)=70bh$
Fractional part $=\frac{9/2}{70}=\boxed{\frac{9}{140}}$.

5. The expression is undefined if $x+2=0$, if $x+4=0$, or if $\frac{x-3}{x+4}=0$. The excluded values are $x=\boxed{-4,-2,3}$.

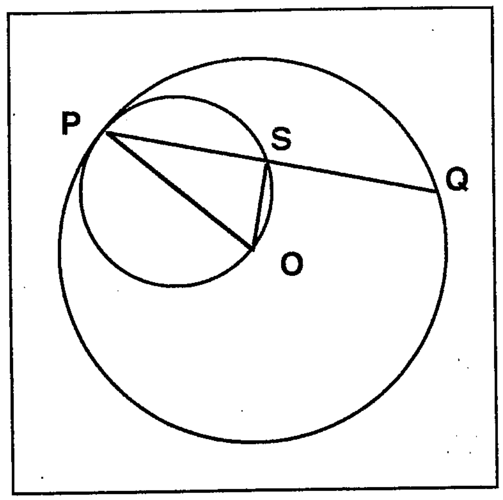
6. In each match, one contestant is eliminated. Since there is only 1 winner, the number eliminated, which equals the required number of matches, is $100-1=\boxed{99}$.

Math Council of Western PA Sr. High Math League 2004 - 2005

Contest # 6 — SOLUTIONS TO TEAM ROUND **March 7, 2005**

T–1. $P(x+c)=(x+c)^3+6(x+c)^2+7(x+c)+2$.
Expanding this, we have
$P(x+c)=x^3+(3c+6)x^2+\left(3c^2+12c+7\right)x+\left(c^3+6c^2+7c+2\right)$.
The coefficient of the x^2 term must be zero, so $3c+6=0 \Rightarrow c=-2$

T–2. Since $\angle OSP$ is a right angle, then $\overline{OS}$ is perpendicular to $\overline{PQ}$. Therefore $\overline{OS}$ bisects $\overline{PQ}$, so $PS=6$.
(NOTE: The length of $\overline{OP}$ does not matter!)



T–3. Since 17 is prime, any positive integral root comes from
$(x-17)(x-1)=x^2-18x+17=0$, so $a=-18$.

T–4. $$\frac{\binom{n}{101}}{\binom{n}{100}}=\frac{n!\ 100!\ (n-100)!}{101!\ (n-101)!\ n!}=\frac{n-100}{101}=k>1.$$ If $k=2$, we have $n=302$.

T–5. Extend $\overline{BC}$ until it reaches $\overline{AE}$. This creates a right triangle with $\overline{BA}$ as a leg and a parallelogram with $\overline{CD}$ as a side. The area of the parallelogram is $4\times 2=8$. Since $AE=5$, the base of the triangle is $5-4=1$. The right triangle has an area of $\frac{1}{2}(3)(1)=1.5$. So, the total area of the pentagon is 9.5 .